Math 126 Summer 2026 HW 3

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Problem 1: Consider the example of a nonlinear diffusion equation

$$\partial_t u - \Delta u^2 = 0 \qquad \{t \geq 0\} \times \mathbb{R}^n$$

Part A.) Assume that $u(t,x) = h(t)\phi(x)$. Find differential equations for v and ϕ (we will solve in parts B. and C).

Part B.) Solve for $v(t)$. If we are given the initial condition $v(0) = a > 0$, what happens to the solution at time a?

Part C.) Assume ϕ is radial and a monomial in the radius, so $\phi(r) = r^\alpha$ for some $\alpha > 0$. What must α be to solve the equation from part A?

Part D.) Form the product solutions.

Problem 2: (Borthwick 5.7 Modified)

The quantum energy levels of a harmonic oscillator in $\mathbb{R}^n$ are the eigenvalues of

$$(-\Delta + |x|^2)\phi = \lambda\phi$$

In the $n = 1$ case, using an exponential term $\phi(x) = q(x)e^{-x^2/2}$ reduces to

$$kqe^{-x^2/2} = -q''e^{-x^2/2} + 2xq'e^{-x^2/2} - x^2qe^{-x^2/2} + q'e^{-x^2/2} + x^2qe^{-x^2/2}$$

or

$$0 = -q'' + 2xq' - (k+1)q$$

Try a solution form $q(x) = \sum_{n=0}^{\infty} a_n x^n$ and derive a relationship for a_{n+2} in terms of a_n (Do not try to get a closed form for the coefficients). For what k does the power series truncate (cut off) to be a polynomial? These are called the Hermite polynomials.

Problem 3: (Integrating the heat kernel) Recall that we used that the integral $\int_{\mathbb{R}^n} H_t(x)dx$ was finite during our proofs. In this exercise, we are going to verify this fact.

Part A.) First, compute the Gaussian integral

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

Hint: Try to instead compute I^2 by writing the integral as $I^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dydx$.

Part B.) Use Part A to compute

$$\int_{\mathbb{R}^n} e^{-|x|^2} dx$$

Part C.) Use a change-of-variables to compute

$$\int_{\mathbb{R}^n} (4\pi t)^{-n/2} e^{-|x|^2/4t} dx$$

Problem 4: (Energy Estimates, Borthwick Problem 6.3 + 6.4) Recall that we used $u(t, x)$ to model the temperature when deriving the heat equation $\partial_t u - \Delta u = 0$. Assuming u is a solution to this equation on $(0, \infty) \times \Omega$ for some bounded domain $\Omega \subset \mathbb{R}^n$, we define the total thermal energy

$$U(t) = \int_\Omega u(t, x)dx$$

Part A.) Assuming $u(t, x)$ satisfies Neumann boundary conditions

$$\frac{\partial}{\partial \nu}|_{\partial\Omega} = 0$$

(for normal vector ν to $\partial\Omega$), show $U(t)$ is constant. Interpret the meaning of this physically.

Part B.) Define the energy of the function

$$\eta(t) = \int_\Omega u(t, x)^2 dx$$

and assume $u(t, x)|_{\partial\Omega} = 0$. Show that η decreases as a function of time.

Part C.) Use B.) to show that a solution $u(t, x)$ satisfying $u|_{t=0} = g$ and $u|_{\partial\Omega} = h$, for g, h continuous on Ω and $\partial\Omega$, us uniquely determined by g and h.